Scientific Computing

Matrix Multiplication

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Outline:

- Basic Definitions, Algorithms and Notation
- Matrix Representation
- Matrix-vector multiplication Ax
  - By inner product
  - By combination of the columns of A
- Matrix-Matrix multiplication
  - By inner product
  - By outer product

Basic Definitions, Algorithms and Notation

Notation: Vector

Let $\mathbb{R}$ designate the set of real numbers. We denote the $\boldsymbol{n}$ real vector $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ as

$$\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \quad \Leftrightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x_i \in \mathbb{R}$$

Vector Operations:

- *transposition* $(\mathbb{R}^{\boldsymbol{n}\times 1} \longrightarrow \mathbb{R}^{1\times \boldsymbol{n}})$,

$$\boldsymbol{y} = \boldsymbol{x}^{\boldsymbol{T}} \quad \Longrightarrow \quad y_i = x_i$$

- *addition* $(\mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{n}})$,

$$\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} \quad \Longrightarrow \quad z_i = x_i + y_i$$

- *scalar-vector, multiplication* $(\mathbb{R} \times \mathbb{R}^{\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{n}})$,

$$\boldsymbol{y} = \alpha \boldsymbol{x} \quad \Longrightarrow \quad y_i = \alpha x_i$$

- *inner product (dot product)* $(\mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{n}} \longrightarrow \mathbb{R})$,

$$a = \boldsymbol{x}^{\boldsymbol{T}} \boldsymbol{y} \Longrightarrow \quad a = \sum_{i=1}^{n} x_i y_i$$

Basic Definitions, Algorithms and Notation

Notation: Matrix

Let $\mathbb{R}$ designate the set of real numbers. We denote the $\boldsymbol{m}$-by-$\boldsymbol{n}$ real matrix $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$ as

$$\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}} \quad \Leftrightarrow \quad A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R}$$

Matrix Operations:

- *transposition* $(\mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{n}\times\boldsymbol{m}})$,

$$C = \boldsymbol{A}^T \quad \Longrightarrow \quad c_{ij} = a_{ji}$$

- *addition* $(\mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}})$,

$$C = \boldsymbol{A} + \boldsymbol{B} \quad \Longrightarrow \quad c_{ij} = a_{ij} + b_{ij}$$

- *scalar-matrix, multiplication* $(\mathbb{R} \times \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}})$,

$$C = \alpha\boldsymbol{A} \quad \Longrightarrow \quad c_{ij} = \alpha a_{ij}$$

- *matrix-matrix, multiplication* $(\mathbb{R}^{\boldsymbol{m}\times\boldsymbol{p}} \times \mathbb{R}^{\boldsymbol{p}\times\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}})$,

$$C = \boldsymbol{AB} \quad \Longrightarrow \quad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Basic Definitions, Algorithms and Notation

Definition: Vector Space on $\mathbb{R}^n$

A **Vector Space on $\mathbb{R}^n$** is a nonempty set of objects, called vectors, subject to the ten axioms listed below. The axioms must hold for all vectors u, v and w in $\mathbb{R}^n$ and for all scalars c and d:

1. The sum of u and v, denoted by $u + v$, is in $\mathbb{R}^n$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a zero vector 0 in $\mathbb{R}^n$, such that $u + 0 = u$
5. For each vector u in $\mathbb{R}^n$, there is a vector $-u$ in $\mathbb{R}^n$, such that $u + (-u) = 0$
6. The scalar multiple of u by c, denoted by cu, is in $\mathbb{R}^n$
7. $c (u + v) = cu + cv$
8. $(c + d) u = cu + du$
9. $c (du) = (cd) u$
10. $1u = u$

Basic Definitions, Algorithms and Notation

Example 1: For all integer $n \geq 0$, $\mathbb{R}^{\boldsymbol{n}}$ is a vector space.

Example 2: $\mathbb{R}$ is a vector space.

Example 3: $\mathbb{R}^{\mathbf{2}}$ is a vector space.

Example 4: Let $M_n(\mathbb{R})$ denotes the set of all $n \times n$ real matrices, then $M_n(\mathbb{R})$ is a vector space.

Example 5: $M_3(\mathbb{R})$ is a vector space.

Basic Definitions, Algorithms and Notation

Definition: Subspace on $\mathbb{R}^n$

Suppose S is a nonempty subset of $\mathbb{R}^n$. We say that S is a ***subspace*** of $\mathbb{R}^n$ if S is a vector space under the addition and scalar multiplication as $\mathbb{R}^n$.

Example 1: Any vector space has two improper subspaces: $\{\mathbf{0}\}$ and the vector space itself. Other subspaces are called proper. The set consisting of only the zero vector, $\{\mathbf{0}\}$, is called the zero subspace.

Example 2: The nonempty set $\boldsymbol{S} = \{(0, y, z) | y, z \in \mathbb{R}\}$ of $\mathbb{R}^3$, is a subspace of vector space $\mathbb{R}^3$.

Example 3: For given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the solution set of the homogeneous linear system $Ax = 0$ is a subspace of $\mathbb{R}^{n\times n}$. This includes all lines, planes, and hyperplanes through the origin.

Example 4: Let $S \in M_n(\mathbb{R})$ denotes the set of all real symmetric $\boldsymbol{n} \times \boldsymbol{n}$ matrices. Then S is a subspace of $M_n(\mathbb{R})$

Basic Definitions, Algorithms and Notation

Theorem: If V is a vector space and S is a *nonempty* subset of V then S is a ***subspace*** of V if and only if S is closed under the addition and scalar multiplication in V.

Example 1: Let $S \in M_n(\mathbb{R})$ denotes the set of real symmetric $\boldsymbol{n} \times \boldsymbol{n}$ matrices. Then S is a subspace of $M_n(\mathbb{R})$:

Proof:

$$S = \{A \in M_n(\mathbb{R}): A^T = A\}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^T$$

1. 0∈S Obvious.

2. If $A, B \in$ S, then $A + B \in$ S:

$$(A+B)^T = A^T + B^T = A + B.$$

3. If $A \in$ S and k is a scaler then $kA \in$ S:

$$(kA)^T = kA^T = kA.$$

$$A = \begin{bmatrix} -2 & 3 & 5 \\ 3 & 4 & -6 \\ 5 & -6 & 7 \end{bmatrix} \Rightarrow$$

$$A^T = \begin{bmatrix} -2 & 3 & 5 \\ 3 & 4 & -6 \\ 5 & -6 & 7 \end{bmatrix} = A$$

Basic Definitions, Algorithms and Notation

Theorem: If V is a vector space and S is a *nonempty* subset of V then S is a ***subspace*** of V if and only if S is closed under the addition and scalar multiplication in V.

Example 2: The nonempty set $\boldsymbol{S} = \{(0, y, z) | y, z \in \mathbb{R}\}$ of $\mathbb{R}^{\mathbf{3}}$, is a subspace of vector space $\mathbb{R}^{\mathbf{3}}$.

1. $(0,0,0) \in$ S Obvious.

2. If $A = (0, y_1, z_1), B = (0, y_2, z_2) \in$ S, then $A + B \in$ S:

$$A + B = (0, y_1, z_1) + (0, y_2, z_2) = (0 + 0, y_1 + y_2, z_1 + z_2) = (0, y, z) \in \mathrm{S}$$

3. If $A = (0, y, z) \in$ S, and k is a scaler then $kA \in$ S:

$$kA = k(0, y, z) = (0k, yk, zk) = (0, y_*, z_*) \in \mathrm{S}$$

Basic Definitions, Algorithms and Notation

Definition: linear combination

Let $v_1, v_2, \ldots, v_k$ a set of vectors in a vector space $\mathbb{R}^n$. A *linear combination* of $v_1, v_2, \ldots, v_k$ is an expression of the form

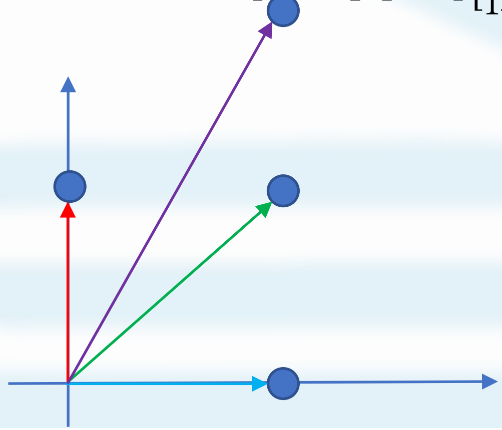
$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

Where $c_1, c_2, \ldots, c_k$ are scalars.

Question:

What's the *linear combination* of $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$?

$$c_1 v_1 + c_2 v_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_1 - c_2 \end{bmatrix} \in \mathbb{R}^2, \quad c_1, c_2 \in \mathbb{R}$$



$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 = 2, c_2 = 0, c_3 = -1$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 =$$

$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + 0 - 1 \\ 2 + 0 + 0 \end{bmatrix}$$

Basic Definitions, Algorithms and Notation

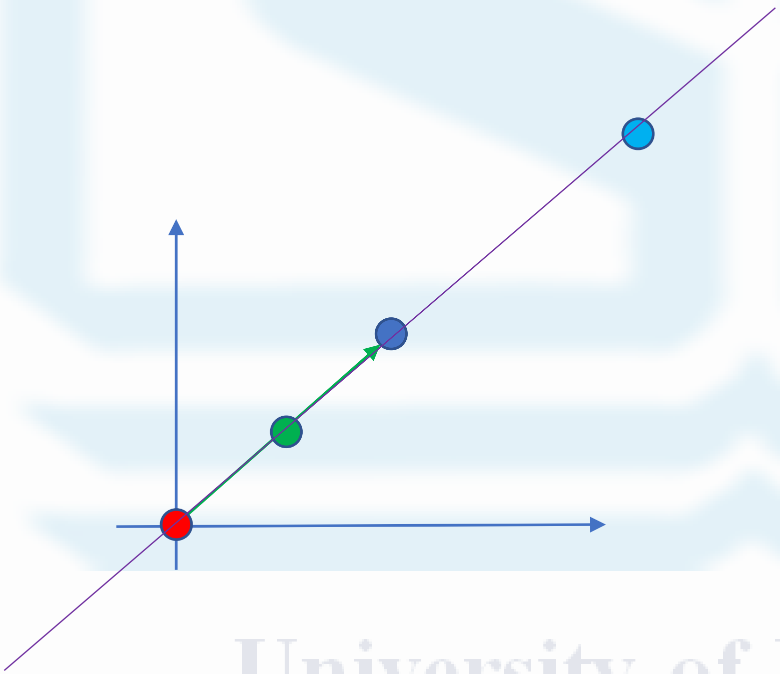
Definition: span

The *span* of $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ is the set of *all linear combinations* of them

$$span\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \colon c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Example 1: The span of a single, nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ is a line through the origin

$$span\{v\} = \{cv \in \mathbb{R}^n \colon c \in \mathbb{R}\}$$



$$c = 0 \;\; ==> \;\; cv = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c = -0.5 \;\; ==> \;\; cv = 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}$$

$$c = 2 \;\; ==> \;\; cv = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\vdots$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Basic Definitions, Algorithms and Notation

Theorem:

Let $v_1, v_2, \ldots, v_k$ a set of vectors in a vector space $\mathbb{R}^{\boldsymbol{n}}$. The span of $v_1, v_2, \ldots, v_k$ is a subspace of $\mathbb{R}^{\boldsymbol{n}}$.

Question:

What's the span of $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^{\mathbf{2}}$?

$$span\{v_1, v_2\} = \{c_1 v_1 + c_2 v_2 \colon c_1, c_2 \in \mathbb{R}\}$$

$$= \left\{ c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \colon c_1, c_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} c_1 + 2c_2 \\ c_1 - c_2 \end{bmatrix} \colon c_1, c_2 \in \mathbb{R} \right\}$$

$$= \mathbb{R}^{\mathbf{2}}$$

Basic Definitions, Algorithms and Notation

Definition: linearly dependent

A set of vectors $v_1, v_2, \ldots, v_k$ from vector space $\mathbb{R}^n$ is said to be ***linearly dependent*** if at least one of the vectors in the set can be defined as a linear combination of the others ; or if there exist scalers c_1, $c_2, \ldots, c_k$, not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = \mathbf{0}.$$

Example.

$$v = \begin{bmatrix}1\\2\end{bmatrix} \quad u = \begin{bmatrix}2\\3\end{bmatrix} \quad t = \begin{bmatrix}5\\8\end{bmatrix} \quad \Longrightarrow \quad t = v + 2v$$

Definition: linearly independent

A set of vectors $v_1, v_2, \ldots, v_k$ from vector space $\mathbb{R}^n$ is said to be ***linearly independent,*** if no vector in the set can be written in this way, then the vectors are said to be *linearly independent*; or if linear combination of, $c_1v_1 + c_2v_2 + \cdots + c_kv_k = \mathbf{0}$, results that all scalers $c_1, c_2, \ldots$ and c_k are zero.

Example.

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix}, \quad v_2 = \begin{bmatrix}2\\3\end{bmatrix}: \quad if \;\; c_1v_1 + c_2v_2 = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = \mathbf{0}$$

Basic Definitions, Algorithms and Notation

Definition: rank of a matrix

The ***rank of a matrix*** is defined as

(a) the maximum number of linearly independent column vectors in the matrix or

(b) the maximum number of linearly independent row vectors in the matrix.

Both definitions are equivalent.

Definition: rank of a matrix

The ***rank of a matrix*** is defined as the number of leading 1s in rref(A).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \Rightarrow rref(A) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Basic Definitions, Algorithms and Notation

Theorem: *If* $A \in \mathbb{R}^{m\times n}$, $P \in \mathbb{R}^{m\times m}$ *and* $Q \in \mathbb{R}^{n\times n}$, P *and* Q *invertible, then*

(a) rank(AQ)=rank(A),
(b) rank(PA)=rank(A),
(c) rank(PAQ)=rank(A).

Corollary: *Elementary row and column operations on a matrix are rank-preserving.*

Matrix Representation

Element wise: $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Row partition: $$A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} \qquad r_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} \in \mathbb{R}^n$$

Column partition: $$A = [c_1 \quad c_2 \quad \cdots \quad c_n] \qquad c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m$$

matrix-vector multiplication Ax

Low Level (for computing):

Inner products of the rows of *A* with *x*

High Level (for understanding):

Combination of the columns of *A*

matrix-vector multiplication
Ax Using ***Inner products***

Low Level (for computing):

Inner products of the rows of *A* with *x*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{bmatrix}$$

$$y_i = \sum_{k=1}^{n} a_{ik} x_j$$

y_i is the inner product of the i^{th} row with the vector x

matrix-vector multiplication

Ax Using *Inner products*

Low Level (for computing):

Inner products of the rows of *A* with *x*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{bmatrix}$$

$$y_1 = \sum_{k=1}^{n} a_{1k} x_j$$

y_1 is the inner product of the *first* row with the vector x

matrix-vector multiplication

Ax Using *Inner products*

Low Level (for computing):

Inner products of the rows of A with x

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{bmatrix}$$

$$y_2 = \sum_{k=1}^{n} a_{2k} x_j$$

y_2 is the inner product of the *second* row with the vector x

matrix-vector multiplication

Ax Using *Inner products*

Low Level (for computing):

Inner products of the rows of *A* with *x*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{bmatrix}$$

$$y_m = \sum_{k=1}^{n} a_{mk} x_j$$

y_m is the inner product of the m^{th} row with the vector x

matrix-vector multiplication

Ax Using *Inner products*

Example:

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{3} a_{1k} x_k \\ \sum_{k=1}^{3} a_{2k} x_k \\ \sum_{k=1}^{3} a_{3k} x_k \end{bmatrix}$$

$$= \begin{bmatrix} [1 \quad 2] \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ [3 \quad 4] \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ [5 \quad 6] \begin{bmatrix} -1 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 19 \end{bmatrix} \coloneqq y$$

$$y_i = \sum_{k=1}^{n} a_{ik} x_j$$

y_i is the inner product of the i^{th} row with the vector x

matrix-vector multiplication
Ax Using *Columns of A*

High Level (for understanding):

Combination of the columns of *A*

matrix-vector multiplication

Ax Using *Columns of A*

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

matrix-vector multiplication

Ax Using *Columns of A*

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$y = A^{(c1)} x_1 +$$

y is a combination of the columns of A

matrix-vector multiplication

Ax Using *Columns of A*

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$y = A^{(c1)}x_1 + A^{(c2)}x_2 +$$

y is a combination of the columns of A

matrix-vector multiplication

Ax Using *Columns of A*

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$y = A^{(c1)}x_1 + A^{(c2)}x_2 + \cdots + A^{(cn)}x_n$$

y is a combination of the columns of A

matrix-vector multiplication

Ax Using *Columns of A*

Example:

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix} + \begin{bmatrix} 8 \\ 16 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 13 \\ 19 \end{bmatrix}$$

y is a combination of the columns of A

$$y = A^{(c1)}x_1 + A^{(c2)}x_2 + \cdots + A^{(cn)}x_n$$

Matrix times Matrix:

- **Inner products**
- **Outer product**

Matrix times Matrix: by *inner products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

c_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix:
by *inner products*

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1p} \\
a_{21} & a_{22} & \cdots & a_{2p} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ip} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mp}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\
b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn}
\end{bmatrix}
=
\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & c_{ij} & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn}
\end{bmatrix}
$$

$$c_{11} = \sum_{k=1}^{p} a_{1k} b_{k1}$$

c_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix:
by *inner products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{21} = \sum_{k=1}^{p} a_{2k} b_{k1}$$

c_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix: by *inner products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{mn} = \sum_{k=1}^{p} a_{mk} b_{kn}$$

c_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix:
by *inner products*

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{3} a_{1k}b_{k1} & \sum_{k=1}^{3} a_{1k}b_{k2} \\ \sum_{k=1}^{3} a_{2k}b_{k1} & \sum_{k=1}^{3} a_{2k}b_{k2} \\ \sum_{k=1}^{3} a_{3k}b_{k1} & \sum_{k=1}^{3} a_{3k}b_{k2} \end{bmatrix}$$

$$= \begin{bmatrix} [1 \quad 2]\begin{bmatrix} -1 \\ 4 \end{bmatrix} & [1 \quad 2]\begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ [3 \quad 4]\begin{bmatrix} -1 \\ 4 \end{bmatrix} & [3 \quad 4]\begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ [5 \quad 6]\begin{bmatrix} -1 \\ 4 \end{bmatrix} & [5 \quad 6]\begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -4 \\ 13 & -6 \\ 19 & -8 \end{bmatrix}$$

c_{ij} is the inner product of the i^{th} row with the j^{th} column

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Matrix times Matrix: by *outer products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

C is a sum of outer products of the columns of A with the rows of B

Matrix times Matrix: by *outer products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$C = A^{(c1)} B_{(r1)} +$$

C is a sum of outer products of the columns of A with the rows of B

Matrix times Matrix:
by *outer products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$C = A^{(c1)}B_{(r1)} + A^{(c2)}B_{(r2)} +$$

C is a sum of outer products of the columns of A with the rows of B

Matrix times Matrix:
by *outer products*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$C = A^{(c1)}B_{(r1)} + A^{(c2)}B_{(r2)} + \cdots + A^{(cp)}B_{(rp)}$$

C is a sum of outer products of the columns of A with the rows of B

Matrix times Matrix: by *outer products*

Example:

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}-1 & 2\\4 & -3\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix}\begin{bmatrix}-1 & 2\end{bmatrix} + \begin{bmatrix}2\\4\\6\end{bmatrix}\begin{bmatrix}4 & -3\end{bmatrix}$$

$$= \begin{bmatrix}-1 & 2\\-3 & 6\\-5 & 10\end{bmatrix} + \begin{bmatrix}8 & -6\\16 & -12\\24 & -18\end{bmatrix}$$

$$= \begin{bmatrix}7 & -4\\13 & -6\\19 & -8\end{bmatrix}$$

C is a sum of outer products of the columns of A with the rows of B

$$C = A^{(c1)}B_{(r1)} + A^{(c2)}B_{(r2)}$$

مراجع

1. Biswa Nath Datta, ***Numerical Linear Algebra and Applications***, 1995.
2. Gilbert Strang, Fourth Edition, ***Linear Algebra and Its Applications***, 2005
3. G.H. Golub, and C.F. Van Loan, **Matrix Computations**, 4rd Edition, Johns Hopkins University Press, 2013.
4. J. Demmel, **Applied Numerical Linear Algebra**, SIAM, 1997.
5. L.N. Trefthen, and lll.D. Bau, **Numerical Linear Algebra**, SIAM, 1997.
6. ***https://math.colorado.edu/~nita/SystemsofLinearEquations.pdf***